

DEFINABILITY LATTICE FOR ADDITION OF RATIONALS

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ABSTRACT. In the present paper we discuss the lattice of reducts of $\langle \mathbb{Q}, \{+\} \rangle$.

1. PRELIMINARIES

We consider the structure $\mathcal{M} = \langle \mathbb{Q}^{<\omega}, \{+\} \rangle$, where $\mathbb{Q}^{<\omega} \subset \mathbb{Q}^{\mathbb{N}}$, $\vec{v} \in \mathbb{Q}^{<\omega}$ if $\{i | v_i \neq 0\}$ is finite. We denote by $\vec{0}$ the vector $\langle 0, \dots, 0, \dots \rangle \in \mathbb{Q}^{<\omega}$. It's well known that \mathcal{M} is ω -saturated elementary extension of $\langle \mathbb{Q}, \{+\} \rangle$, so the lattice of definable reducts of \mathcal{M} (and $\langle \mathbb{Q}, \{+\} \rangle$) corresponds to the lattice of subgroups of the group of permutations $Sym(\mathcal{M})$, containing the group $GL(\mathcal{M})$ of invertible linear maps.

Our consideration consists from 3 parts.

Dyadic relations. Here we discuss dyadic relations and 2-definable relations – relations, definable by the signature $\{y = rx | r \in \mathbb{Q}\}$. These relations are almost trivial, but form rather complicated infinite lattice.

Triadic relations. Roughly speaking triadic relations add 2 new reducts: $z = (x + y)/2$ and $z = \pm x \pm y$ (by $y = \pm x_1 \dots \pm x_n$ we denote the relation $\bigvee_{s \in \{-1,1\}^n} y = \sum_{i=1}^n s(i)x_i$). In particular we reprove, that the group $AGL(\mathcal{M})$ is maximal ([1]).

Relations with more then 3 arguments. We show that they add no new reducts.

If a relation $R(x_1, \dots, x_n)$ is definable in \mathcal{M} , tuples a_1, \dots, a_n and b_1, \dots, b_n are linearly independent, then $R(\vec{a}) \equiv R(\vec{b})$, so we will suppose, that $\neg R(\vec{a})$ for a linearly independent tuples \vec{a} . In other words we suppose that $\{\sum_{i=1}^n r_i x_i \neq \vec{0} | r_i \in \mathbb{Q}, r_i \neq 0 \text{ for some } i\} \cup \{R(x_1, \dots, x_n)\}$ is inconsistent, so

Note 1. $R(x_1, \dots, x_n) \rightarrow \bigvee_{j=1}^K \sum_{i=1}^n r_{j,i} x_i = \vec{0}$ for some K .

From now on a definable relation is a relation, definable in the structure $\langle \mathcal{M}, \{+\} \rangle$, definable by a signature Σ means definable in the structure $\langle \mathcal{M}, \Sigma \rangle$; independent tuple \vec{t} is a linearly independent $\vec{t} = \langle t_1, \dots, t_n \rangle, t_i \in \mathcal{M}$; by $l(a, b), a, b \in \mathcal{M}$ we denote the straight line passing through a and b .

We say, that a tuple a_1, \dots, a_n is m -independent for some natural m if $\sum_{i=1}^n (k_i/l_i)a_i = \vec{0}, |k_i| < m, |l_i| < m$ implies all $k_i = 0$.

Due to standard compactness arguments we note, that

Note 2. For any definable relation $R(\bar{x})$ exists such natural number m that $R(\bar{x})$ holds for any independent \bar{x} iff $R(\bar{x})$ holds for any m -independent \bar{x}

We use abbreviations $(\exists_{>k}x)Q(x)$ for $(\exists x_1, \dots, x_{k+1})(\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{i=1}^n Q(x_i))$, $(\exists_{<k}x)Q(x)$ for $(\exists x)Q(x) \wedge \neg(\exists_{>k-1}x)Q(x)$, and $(\exists_{=k}x)Q(x)$ for $(\exists_{>k-1}x)Q(x) \wedge \neg(\exists_{>k}x)Q(x)$. Sometimes we use the abbreviation $(\exists_{>k}x_1, \dots, x_n)$ which is defined by induction: $(\exists_{>k}x_1, \dots, x_n)Q(x_1, \dots, x_n) \Leftrightarrow (\exists_{>k}x_1)((\exists_{>k}x_2, \dots, x_n)Q(x_1, \dots, x_n))$.

Note 3. For a definable relation $R(\bar{x}, \bar{y})$ there exists a natural number K such that holds

$$\neg(\exists_{>K}\bar{y})R(\bar{x}, \bar{y}) \vee \neg(\exists_{>K}\bar{y})\neg R(\bar{x}, \bar{y})$$

Proof. This is shown by contradiction. Suppose that $(\exists x)((\exists_{>K}\bar{y})R(\bar{x}, \bar{y}) \wedge (\exists_{>K}\bar{y})\neg R(\bar{x}, \bar{y}))$ holds for any K . Then the set $\{R(\bar{a}, \bar{b}), \neg R(\bar{a}, \bar{b}')\} \cup \{\sum_i p_i a_i + \sum q_i b_i \neq \bar{0} | p_i, q_i \in \mathbb{Q}, q_i \neq 0 \text{ for some } i\} \cup \{\sum_i p_i a_i + \sum q_i b'_i \neq \bar{0} | p_i, q_i \in \mathbb{Q}, q_i \neq 0 \text{ for some } i\}$ is consistent, hence due to ω -saturation of \mathcal{M} there are $\bar{a}, \bar{b}, \bar{b}' \in \mathcal{M}$ such that \bar{b}, \bar{b}' are independent, $\mathcal{V}(\bar{a}) \cap \mathcal{V}(\bar{b}) = \mathcal{V}(\bar{a}) \cap \mathcal{V}(\bar{b}') = \{\bar{0}\}$, and $R(\bar{a}, \bar{b}), \neg R(\bar{a}, \bar{b}')$. Contradiction, because there is $\sigma \in GL(\mathcal{M})$ such that $\sigma(\bar{a}) = \bar{a}, \sigma(\bar{b}) = \bar{b}'$. \square

So the note 2 can be reformulated as

Corollary 1. For any definable relation $R(\bar{x}, \bar{y})$ exists a natural number K such that $R(\bar{a}, \bar{b})$ holds for some (any) independent \bar{b} such that $\mathcal{V}(\bar{a}) \cap \mathcal{V}(\bar{b}) = \{\bar{0}\}$ iff $(\exists_{>K}\bar{y})R(\bar{a}, \bar{y})$ holds.

Note, that there is only one nontrivial definable subset of \mathcal{M} , i.e. $\{\bar{0}\}$.

2. DYADIC RELATIONS

Statement 1. If $R(x_1, \dots, x_n)$ is nontrivial 2-definable relation, then $\{\bar{0}\}$ is definable by R .

Proof. We may suppose that $R(a_1, \dots, a_n) \Rightarrow a_i \neq a_j$.

The rank of a relation $R(x_1, \dots, x_n)$ is a maximum number m , such that $R(a_1, \dots, a_n)$ holds for a tuple \bar{a} , containing m independent items. Note, that we consider relations $R(x_1, \dots, x_n)$ which rank is less than n .

Let m be the rank of R . Then (renumbering variables if necessary) $R(a_1, \dots, a_m, b_1, \dots, b_{n-m})$ holds for the independent tuple \bar{a} and some tuple \bar{b} . The relation R is 2-definable and m is the rank of R , so (1) each $b_j \in l(\bar{0}, a_i)$ for some i and (2) $\{\bar{b}' | R(\bar{a}, \bar{b}')\}$ is finite (Note 3). We suppose, that $b_1 \in l(\bar{0}, a_1)$ hence for some $k \neq 0$ and sufficiently large M holds $(\exists_{=k}y_1)(\exists_{>M}x_2, \dots, x_m)(\exists y_2, \dots, y_{n-m})(y_1 \neq a_1 \wedge R(a_1, x_2, \dots, x_m, y_1, \dots, y_{n-m}))$. Denote by $Q(x, y)$ the statement $(\exists_{>M}x_2, \dots, x_m)(\exists y_2, \dots, y_{n-m})R(x, x_2, \dots, x_m, y, \dots, y_{n-m})$. We see that $|\{c | Q(a_1, c), c \neq a_1\}| = k$. Therefore $|\{c | Q(d, c), c \neq d\}| = k$ for any $d \neq \bar{0}$. But $\{c | P(\bar{0}, c), c \neq \bar{0}\}$ is or empty or infinite for any definable P . \square

Let $R(x, y)$ is definable relation, we may suppose that $(\exists x)(\exists y)(R(x, y) \wedge x \neq y \wedge x \neq \vec{0} \wedge y \neq \vec{0})$ – otherwise it's equivalent to a definable subset of \mathcal{M} .

According to the note 1 a $R(x, y) \wedge x \neq \vec{0}$ is equivalent to $\bigvee_{i=1}^n y = r_i x$ for some $r_1, \dots, r_n, r_i \neq 1, 0$.

Denote by G the multiplicative group generated by set $\{r_1, \dots, r_n\}$, and define the equivalence relation \sim on \mathcal{M} such, that $a \sim b \iff a = rb$ for some $r \in G$. A permutation $\varphi: \mathcal{M} \rightarrow \mathcal{M}$, preserving the relation R , is a composition of a permutation on \mathcal{M}/\sim and bijections between corresponding classes of the equivalence.

For each $a \in \mathcal{M}$ there is a corresponding permutation σ_a on the set $\{r_1, \dots, r_n\}$, such that $\varphi(a \cdot r_i) = \varphi(a) \cdot \sigma_a(r_i)$. These permutations σ_a describe the corresponding bijections.

If $a \approx b$, then permutations σ_a and σ_b are independent. If $a \sim b$, then permutations σ_a and σ_b are nearly the same.

Statement 2. *If $a \sim b$, then $|\sigma_a(r_i)| = |\sigma_b(r_i)|$.*

Proof. It's enough to show, that for any $a \in \mathcal{M}, r_i, r_j$ holds $|\sigma_a(r_j)| = |\sigma_{a \cdot r_i}(r_j)|$. We enumerate r_1, \dots, r_n such, that $|\sigma_a(r_1)| \leq |\sigma_a(r_2)| \leq \dots, |\sigma_a(r_n)|$.

By induction on $m \leq n$ we prove that

$$|\sigma_{a \cdot r_i}(r_j)| = |\sigma_a(r_j)|; |\sigma_{a \cdot r_j}(r_i)| = |\sigma_a(r_i)| \text{ for all } i \leq m, j \leq n$$

For a current m we need to prove that $|\sigma_{a \cdot r_m}(r_j)| = |\sigma_a(r_j)|$ and $|\sigma_{a \cdot r_j}(r_m)| = |\sigma_a(r_m)|$ for all $j \geq m$.

The proof is by induction on j .

First we show, that $|\sigma_{a \cdot r_m}(r_j)| = |\sigma_a(r_j)|$. Suppose not, so $\sigma_{a \cdot r_m}(r_i) = \sigma_a(r_j)$ for some $r_i, |\sigma_a(r_i)| \neq |\sigma_a(r_j)|$. Then $i > j$, because if $i < j$ then, by the induction hypothesis, $|\sigma_{a \cdot r_m}(r_i)| = |\sigma_a(r_i)|$. Then $\varphi(a) \cdot \sigma_a(r_m) \cdot \sigma_a(r_j) = \varphi(a \cdot r_m \cdot r_i) = \varphi(a) \cdot \sigma_a(r_i) \cdot \sigma_{a \cdot r_i}(r_m)$, i.e. $\sigma_a(r_m) \cdot \sigma_a(r_j) = \sigma_a(r_i) \cdot \sigma_{a \cdot r_i}(r_m)$. Because $|\sigma_a(r_i)| > |\sigma_a(r_j)|$, then $|\sigma_{a \cdot r_i}(r_m)| < |\sigma_a(r_m)|$, and $|\sigma_{a \cdot r_i}(r_m)| = |\sigma_a(r_k)|$ for some $k < m$. But according the induction hypothesis $|\sigma_{a \cdot r_i}(r_k)| = |\sigma_a(r_k)|$ holds for all $k < m, i < n$. Contradiction.

Show now that $|\sigma_{a \cdot r_j}(r_m)| = |\sigma_a(r_m)|$. Note that $\varphi(a \cdot r_m \cdot r_i) = \varphi(a) \cdot \sigma_a(r_m) \cdot \sigma_{a \cdot r_m}(r_j) = \varphi(a) \cdot \sigma_a(r_j) \cdot \sigma_{a \cdot r_j}(r_m)$. Because it was already shown that $|\sigma_{a \cdot r_m}(r_j)| = |\sigma_a(r_j)|$, we conclude, that $|\sigma_{a \cdot r_j}(r_m)| = |\sigma_a(r_m)|$.

End of induction on j .

End of induction on m . □

Statement 3. *If $|r_i| = |r_j|$ then $|\sigma_a(r_i)| = |\sigma_a(r_j)|$*

Proof. $|\varphi(a \cdot r_i \cdot r_i)| = |\varphi(a)| \cdot |\sigma_a(r_i)| \cdot |\sigma_{a \cdot r_i}(r_i)| = |\varphi(a)| \cdot |\sigma_a(r_i)| \cdot |\sigma_a(r_i)| = |\varphi(a \cdot r_j \cdot r_j)|$ □

2.1. Dyadic relations summary. Automorphism groups of definable dyadic relations closely connected with automorphism groups of finitely generated abelian groups (e.g. [3]).

First of all we describe a group G_R of automorphisms for a relation $R(x, y) \equiv \bigvee_{i=0}^{n-1} x = r_i y$ where $|r_i| \neq |r_j|$. Let G be the multiplicative group generated by set $\{r_1, \dots, r_n\}$. We say that permutation σ on the set $\{r_1, \dots, r_n\}$ is *correct* if the mapping $\psi(r_1^{k_1} \dots r_n^{k_n}) = \sigma(r_1)^{k_1} \dots \sigma(r_n)^{k_n}$ is an automorphism of G , in other words if σ preserves all multiplicative dependences between $\{r_1, \dots, r_n\}$.

A permutation φ , preserving the relation R on the structure \mathcal{M} is the composition of a permutation on \mathcal{M}/\sim and the permutations φ_f of group G for each bijection f between corresponding classes of the equivalence. Each permutation φ_f corresponds to an automorphism generated by a correct permutation on the set $\{r_1, \dots, r_n\}$.

A relation R' is definable by a relation R if $G_R \subset G'_R$, i.e. if all r'_i belongs to the group, generated by the set $\{r_1, \dots, r_n\}$ and each correct permutation on $\{r_1, \dots, r_n\}$ generates (correct) permutation on $\{r'_1, \dots, r'_{n'}\}$.

A group G_R is a bit more complicated when $|r_i| = |r_j|$ for some i, j . Denote by $P = \{s_1, \dots, s_k\}$ subset of such numbers from $\{r_1, \dots, r_n\}$ that $-s_i \notin \{r_1, \dots, r_k\}$ and by G' the multiplicative group generated by P . Let ϕ be a mapping of cosets of G' in G to $\{-1, 1\}$. Then a permutation φ_f of group G , corresponding to a bijection between corresponding classes of the equivalence is $\phi(x)\sigma(x)$ where σ is an automorphism, generate by a correct permutation on the set $\{r_1, \dots, r_n\}$ and ϕ .

We call a relation R *2-(un)definable* if it's (un)definable by dyadic relations, i.e. is (un)definable by the signature $\{y = rx | r \in \mathbb{Q}\}$.

By $LGL(\mathcal{M})$ we denote the group of permutations of \mathcal{M} , preserving all 2-definable relations. We note that $LGL(\mathcal{M}) = \langle GL(\mathcal{M}), Sym_l \rangle$, where Sym_l is the group of permutations on the set of straight lines, passing through $\vec{0}$.

3. TRIADIC RELATIONS.

According the note 1 holds $R(x, y, z) \rightarrow a_1x + b_1y + c_1z = \vec{0} \vee \dots \vee a_nx + b_ny + c_nz = \vec{0}$. So there are expressions $a_1x + b_1y + c_1z = \vec{0}, \dots, a_kx + b_ky + c_kz = \vec{0}$ such, that $R(x, y, z) \equiv a_1x + b_1y + c_1z = \vec{0} \vee \dots \vee a_kx + b_ky + c_kz = \vec{0}$ for any linearly independent x, y . We can suppose, due to linearly independence of x, y , that $c_i \neq 0$, so we can rewrite the equations in the form $R(x, y, z) \equiv z = p_1x + q_1y \vee \dots \vee z = p_kx + q_ky$. The list of expressions $z = p_1x + q_1y, \dots, z = p_kx + q_ky$ or simply the list of pairs $\langle p_1, q_1 \rangle, \dots, \langle p_k, q_k \rangle$ we will call the *table* of relation R .

First of all we simplify the relation R .

Consider the relation $R'(x, y, z) \equiv R(x, y, z) \wedge \neg(\exists_{>K} y') R(x, y', z)$ for sufficiently large natural number K (Note 3). It's easy to see, that (i) the table of R' is a subset of the table of R , and (ii) $q_i \neq 0$ for all q_i from the table of R' . So we can remove from the table of R all expressions where $p_i = 0$ or $q_i = 0$. If we removed all expressions from the table of R it means that R is 2-definable, so from now on we

suppose that $p_i \neq 0, q_i \neq 0$ for all expressions in the table. The process of removing lines where $p_i = 0$ or $q_i = 0$ we'll call *normalization*, the result of normalization is *normal form* of relation.

Now we can suppose that for all independent x, y holds $|\{z | R(x, y, z)\}| < k$, so for dependent x, y holds $R(x, y, z) \rightarrow z \in l(x, y)$, where $l(x, y)$ is the line passing through x, y . Otherwise we consider the relation $R'(x, y, z) \Leftarrow R(x, y, z) \wedge (\exists_{<_{k+1}} z') R(x, y, z')$.

We call the relation $R(x, y, z)$ *affine* if $p_i + q_i = 1$ for all pairs from the table of R . In other word R is affine if for any independent x, y holds $R(x, y, z) \Rightarrow z \in l(x, y)$.

The group $AGL(\mathcal{M})$ is the group of affine permutations: it contains, beside $GL(\mathcal{M})$, permutations $\sigma(x) = x + v$ for all $v \in \mathcal{M}$.

We are going to prove:

Lemma 1. *If R is affine, and permutation σ preserves R then $\sigma \in AGL(\mathcal{M})$.*

Corollary 2. *If $\vec{0}$ is definable by an affine relation R , then R is equivalent (as reduct) to $z = x + y$.*

If $\vec{0}$ is not definable by an affine relation R , then R is equivalent (as reduct) to $z = (x + y)/2$.

Lemma 2. *$\vec{0}$ is definable by any nonaffine relation R .*

Statement 4. *Nonaffine relation R is*

(i) *equivalent to $z = x + y$*

or

(ii) *is equivalent to $\{z = \pm x \pm y, R^*\}$ for some 2-definable relation R^* .*

Proof. **lemma 1.**

First we prove

Lemma 3. *By R can be defined a relation $S(x, y, z)$, such that*

(i) *$\{z | S(x, y, z)\} \subset l(x, y)$ and*

(ii) *for independent x, y holds $S(x, y, (x + y)/2)$.*

Proof. **lemma 3**

We say that a relation $S(x, y, z)$ is *r-correct*, if for any independent x, y condition (i) holds and $S(a, b, a + r(b - a))$. The relation R is *q-correct* for some rational $q, q \neq 0, q \neq 1$. Let us show, that by *r-correct* relation $S(x, y, z)$ can be defined $1/r$ and $r/(1 - r)$ correct relations. First, it's easy to see that $S(x, z, y)$ is $1/r$ -correct.

Second, note that the relation $(\exists v)(S(x, v, z) \wedge S(v, x, y))$ is $r/(1 - r)$ -correct. Condition (i) follows from the condition (i) for S . We need to show, that if a, b are independent, then $(\exists v)(S(a, v, c) \wedge S(v, a, b))$ where $c = a + (r/(1 - r))(b - a)$. Let $v = b + (c - a)$. It's easy to check, that $S(a, v, c) \wedge S(v, a, b)$.

Using operations $r \rightarrow 1/r, r \rightarrow r/(1-r)$ we can from any q -correct relation build a $1/2$ -correct relation. **End of lemma 3 proof.** \square

According to *the fundamental theorem of affine geometry* ([2]) a permutation σ of \mathcal{M} belongs to $AGL(\mathcal{M})$ iff σ takes any 3 collinear points to 3 collinear points.

So we are going to show, that if σ preserves the relation R , then it takes any 3 collinear points to 3 collinear points. We start with nonzero points. Suppose, that a, b, c – 3 collinear points, $a, b, c \neq \vec{0}, c = a + r(b - a), 0 < r < 1$. Let $S(x, y, z)$ be definable by R $1/2$ -correct relation. Because $S(a, b, (a + b)/2)$ holds for any independent a, b , it must holds for a n -independent a, b when n is sufficiently large. Choose such small vector Δ , that $b = a + k\Delta, c = a + l\Delta$ for some integers $k, l, k > l$ and pairs $a + i\Delta, a + (i + 2)\Delta$ are n -independent for $i < k$. Then holds

$$(1) \quad (\exists x_0, \dots, x_k)(x_0 = a \wedge x_k = b \wedge x_l = c \wedge \bigwedge_{i=1}^{k-2} S(x_i, x_{i+2}, x_{i+1}))$$

– we can set $x_i = a + i\Delta$. Permutation σ has to preserve S as well as equation (1). From the condition (i) follows, that $\sigma(x_i)$ has to lie on the same line.

So the permutation σ takes any 3 collinear nonzero points to 3 collinear points. Show now that the point $\vec{0}$ keeps collinearity as well.

To the contrary. For any line $l, \vec{0} \in l$ by the $\sigma'(l)$ we denote the line, containing all points from $\sigma(l \setminus \{\vec{0}\})$. Suppose that $\vec{0}$ lies on line l but $\sigma(l)$ does not contain the point $\sigma(\vec{0})$.

Consider 2 cases. (i) $\sigma(\vec{0}) \neq \vec{0}$. Take some nonzero point a on the line l , consider a line l' , passing through $\sigma(\vec{0}), \sigma(a)$. Inverse images of all nonzero points of line l' lie on the line l and inverse images of all nonzero points of line $\sigma'(l)$ lie on the line l . Contradiction.

Case (ii). $\sigma(\vec{0}) = \vec{0}$. Choose another line l' passing through $\vec{0}$. Lines $\sigma'(l)$ and $\sigma'(l')$ don't intersect. Take an arbitrary line l'' , parallel to l . Lines $\sigma'(l), \sigma'(l'')$ are parallel, which contradict the intersection of $\sigma'(l')$ and $\sigma'(l'')$.

End of lemma 1 proof. \square

Proof. Proof of corollary 2.

Let R be an affine relation. The group of permutation, preserving R is a subgroup of $AGL(\mathcal{M})$. If $\vec{0}$ is definable by R , then this subgroup preserves $\vec{0}$, so it coincides with $GL(\mathcal{M})$. If $\vec{0}$ is not definable by R , then it contains a shift $x \rightarrow x + v$ for nonzero v . In this case it coincides with $AGL(\mathcal{M})$. \square

To prove the lemma 2 we need that p, q satisfy conditions: $p^2 + q \neq 0; p + q^2 \neq 0; p \neq q$.

So it may be necessary to transform the relation R .

Lemma 4. *For any nonaffine relation R there is a relation $R'(x, y, z)$, definable by R which table contains a line $z = px + qy$, where $p, q \neq 0; p^2 + q \neq 0; p + q^2 \neq 0; p \neq q, p + q \neq 1$.*

Proof. If $p = -1, q = -1$ then we consider the relation $Q(x, y, u, z) \Leftarrow (\exists v_1, v_2, v_3, v_4)(R(x, y, v_1) \wedge R(x, u, v_2) \wedge R(y, u, v_3) \wedge R(v_1, v_2, v_4) \wedge R(v_4, v_3, z))$. For independent x, y, u it holds when $v_1 = -x - y; v_2 = -x - u; v_3 = -y - u; v_4 = -v_1 - v_2; z = -v_4 - v_2 = -(-(-x - y) - (-x - u)) - (-y - u) = -2x$. Take the relation $S(x, z) \Leftarrow (\exists_{>M} y)(\exists_{>M} v)Q(x, y, v, z)$ for sufficiently large M (Corollary 1). The set $\{z | S(x, z)\}$ is finite and contains $-2x$ for a nonzero x . So the table of relation $R'(x, y, z) \Leftarrow (\exists x')(\exists y')(S(x, x') \wedge S(y, y') \wedge R(x', y', z))$ contains the item $z = 2x + 2y$.

If $p = q$, then choose the relation $(\exists z')(R(x, y, z') \wedge R(z', y, z))$ which contains the line $p^2x + q(p+1)y$.

If $p + q^2 = 0$ or $p^2 + q = 0$, then we consider a sequence $R_0 \Leftarrow R, R_{i+1}(x, y, z) \Leftarrow (\exists z_1, z_2)(R_i(x, y, z_1) \wedge R_i(y, x, z_2) \wedge R_i(z_1, z_2, z))$ of relations. The table of R_{i+1} contains the line $(p_i^2 + q_i^2)x + 2p_iq_iy = z$ for any $p_ix + q_iy = z$ from R_i . Hence the table of R_k for sufficiently large k contains a line $p_kx + q_ky = z$ where $p_k, q_k \neq 0; p_k^2 + q_k \neq 0; p_k + q_k^2 \neq 0; p_k \neq q_k, p_k + q_k \neq 0$. \square

Proof. lemma 2.

Suppose, that R is nonaffine relation, i.e. $p + q \neq 1$ holds for some item $px + qy = z$ of the table of R .

We are going to prove that $\vec{0}$ is definable by R . To the contrary.

Due to lemma 4 we suppose, that $p, q \neq 0; p^2 + q \neq 0; p + q^2 \neq 0; p \neq q$.

We define relations $R_1(x, y, z) \Leftarrow (\exists z')(R(x, y, z') \wedge R(y, z', z)); R_2(x, y, z) \Leftarrow (\exists z')(R(y, x, z') \wedge R(z', y, z))$. Due to normalization we can suppose that tables R_1, R_2 contains no zero items.

We claim, that for some $K > 0$, sufficiently large M , and for any $b \neq \vec{0}$ holds $(\exists_{<K} v, v \neq b)(\exists_{>M} u)(\exists w)(R_1(u, b, w) \wedge R_2(u, v, w))$.

First note that the line $pqx + (p + q^2)y$ is in the R_1 table, and the line $pqx + (q + p^2)y$ is in the R_2 table.

Take $b \neq \vec{0}$, and denote $S = \{c | c \neq b, (\exists_{>M} u)(\exists w)(R_1(u, b, w) \wedge R_2(u, c, w))\}$ where M is sufficiently large, as in Note 3.

Show that the set S is nonempty. Choose $c = b(p^2 + q)/(q^2 + p)$. It's easy to see that $c \neq b$, we will prove that $(\exists_{>M} u)(\exists w)(R_1(u, b, w) \wedge R_2(u, c, w))$. It's enough to demonstrate that $(\exists w)(R_1(a, b, w) \wedge R_2(a, c, w))$ holds for any a , which is independent with b . For this we can set $w = pqa + (q + p^2)b$.

The set S can not be too big. Suppose $c \in S$. Because M is sufficiently large, so $(\exists w)(R_1(a, b, w) \wedge R_2(a, c, w))$ holds for any $a \notin \mathcal{V}(\{b, c\})$. It means that $q_2c = q_1b$ for some q_1, q_2, p_0 , such that $p_0x + q_1y$ is in the R_1 table, and $p_0x + q_2y$ is in the R_2 table. Because all table constants are nonzero, there are fixed number of such c .

We supposed that $\vec{0}$ is undefinable, so $(\exists_{<K} v, v \neq \vec{0})(\exists_{>M} u)(\exists w)(R_1(u, \vec{0}, w) \wedge R_2(u, v, w))$ has to hold. Contradiction. **End of lemma 2 proof.** \square

Due to lemma 2 from now we consider $\vec{0}$ as the symbol of the signature.

We call a relation $R(x, y, z)$ *simple*, if sentence $R(a, b, c) \Leftrightarrow (c = pa + qb \text{ for some table line } px + qy = z)$ holds not only for independent a, b but also for any nonzero a, b, c .

Define $R'(x, y, \Delta, z) \Leftarrow (\exists v)(R(x, \Delta, v) \wedge R(v, y, z))$, R'' is the normal form of $R'(x, y, \Delta, z)$. Now denote by R_M^* the normal form of $(\exists_{>M} \Delta)(\exists w)(R'(x, y, \Delta, w) \wedge R''(z, z, \Delta, w))$. The relation R_M^* we call *simplification* of R .

Lemma 5. *For any relation R there is a simple definable by R relation.*

Proof. Due to lemma 4 we suppose that in the table of R there is a pair such that $p, q \neq 0; p^2 + q \neq 0; p + q^2 \neq 0; p \neq q$.

We show that for sufficiently large M the simplification R_M^* of R is simple.

Note that the table of R_M^* contains lines $(p_1 p_2 x + q_1 y) / (p_3 p_4 + q_3)$ where $p_i x + q_i y, i = 1, 2, 3, 4$ are (not necessarily different) lines of R and $p_1 q_2 = p_3 q_4, p_3 p_4 + q_3 \neq 0$.

Take $a, b, c \neq \vec{0}$. Prove that for sufficiently large M holds $R_M^*(a, b, c) \Leftrightarrow (p^* a + q^* b = c \text{ for some } p^*, q^* \text{ from table } R^*)$

(i) \Leftarrow . $c = (p_1 p_2 a + q_1 b) / (p_3 p_4 + q_3)$ for some 4 lines of R , such, that $p_1 q_2 = p_3 q_4$. Choose a vector $\Delta \notin \mathcal{V}(\{a, b, c\})$. Then all pairs $\{a, v\}, \{v, b\}, \{v', c\}$ are independent, $R(a, \Delta, v)$ and $R(c, \Delta, v')$ holds, so $R_M^*(a, b, c)$ holds as well.

(ii) \Rightarrow . To the contrary. Suppose that $p^* a + q^* b \neq c$ for all lines of table R_M^* , but $(\exists_{>M} \Delta)(\exists w)(R'(a, b, \Delta, w) \wedge R'(c, c, \Delta, w))$. The M is sufficiently large to ensure that there is a vector $\Delta \notin \mathcal{V}(\{a, b, c\})$ and $R'(a, b, \Delta, w) \wedge R'(c, c, \Delta, w)$ holds (Corollary 1). From independency follows that $w = p_1 q_2 \Delta + p_1 p_2 a + q_1 b = p_3 q_4 \Delta + (p_3 p_4 + q_4) c$, where $p_3 p_4 + q_4 \neq 0$. And again from independency $p_1 q_2 = p_3 q_4, c = (p_1 p_2 a + q_1 b) / (p_3 p_4 + q_3)$ – contradiction. \square

Lemma 6. *The relation $z = \pm x \pm y$ is definable by any simple relation.*

Proof. Let R be simple relation. The proof of lemma 6 consists from few steps.

Step 1. *Denote be Q the set of second components of the table $\{(p_i, q_i)\}$ of relation R .*

(i) *There is a relation definable by R which table is $P \times Q$ for some nonempty P .*

(ii) *The relation $D(y_1, y_2) \Leftrightarrow y_1 = (q_i / q_j) y_2, q_i, q_j \in Q$ is definable by R .*

Proof. Let N be the number of items in the table R . Define dyadic relations $W_1(x, z) \Leftarrow (x \neq \vec{0}) \wedge (\exists_{<N} y, y \neq \vec{0}) R(x, y, z), W_2(y, z) \Leftarrow (y \neq \vec{0}) \wedge (\exists_{<N} x, x \neq \vec{0}) R(x, y, z)$. Note that for independent \vec{a}, \vec{b} holds $\neg W_1(\vec{a}, \vec{b})$ because each table line $p_i \vec{a} + q_i y = \vec{b}$ has a nonzero solution in y and all solutions are different. Hence $\{z | W_1(\vec{a}, z)\}$ is a finite subset of $l(\vec{0}, \vec{a})$ for $\vec{a} \neq \vec{0}$. From the other hand $W_1(\vec{a}, p_i \vec{a})$ holds for any $\vec{a} \neq \vec{0}$ and p_i from the table R because $p_i \vec{a} + q_i y = p_i \vec{a}$ has no nonzero solution.

Consider a simple relation $R_1(x, y, z) \Leftarrow (\exists v)(W_1(v, x) \wedge R(v, y, z))$.

Note that table of R_1 contains the line $x + q_i y$ for each $q_i \in Q$. Define a relation $R_2(x, y, z) \Leftarrow R_1(x, y, z) \wedge (\exists_{=K} v)(y \neq v \wedge R(x, v, z) \wedge (\exists w)(W_2(w, y) \wedge W_2(w, v)))$, where K is the number of items in Q .

We show, that for independent x, y holds $R_2(x, y, z) \Leftrightarrow (p_i x + q_i y = z \text{ for } \{p_i, q_i\} \text{ from the table } R_1 \text{ such that } \{p_i, q_i\} \text{ belongs to the table of } R_1 \text{ for any } q \in Q)$.

\Rightarrow . If $R_2(x, y, z)$ holds then $p_i x + q_i y = z$ for some $\{p_i, q_i\}$. If $(\exists w)(W_2(w, y) \wedge W_2(w, v))$ holds then $v = ry$ for some $r \in Q$. Hence there are K different rational numbers r_1, \dots, r_K such that $p_j x + q_j r_m y = z$ holds for some $\{p_j, q_j\}$. From the independency if x, y follows that $p_j = p_i$ and all q_j are different.

\Leftarrow . Let $p_i x + q_i y = z$ holds for some p_i , such that $\{p_i, q_i\}$ belongs to the table of R_1 for any $q \in Q$. Note that $(\exists w)(W_2(w, y) \wedge W_2(w, v))$ holds for any $v = (q/q_i)y, q \in Q$ so $R_2(x, y, z)$ holds.

The table of R_2 is not empty because $R_2(x, y, x + q_i y)$ holds for any $q_i \in Q$.

Define $D(y_1, y_2) \Leftarrow (\exists_{>M} x, z)(R^*(x, y_1, z) \wedge R^*(x, y_2, z))$ for sufficiently large M . \square

Step 2. *There is a relation definable by R which table contains lines of form $a(x - y) = z$ or $a(x \pm y) = z$ only.*

Proof. If $R_2(x, y, z)$ holds for independent x, y then $S_{x, y, z} = \{y' | R_2(x, y', z)(\exists w)(W_2(w, y) \wedge W_2(w, y'))\}$ consists of items $qy/q_i, q \in Q$, where $z = p_i x + q_i y$. So $S_{x, y, z_1} = S_{x, y, z_2}$ if $z_1 = p_1 x + q_1 y, z_2 = p_2 x + q_2 y$ and or $q_1 = q_2$ or $q_1 = -q_2$ (in the latter case Q has to contain $-q$ for any $q \in Q$).

Consider the relation $R_3(x, y, z) \Leftarrow (\exists v)(S_{z, v, x} = S_{z, v, y})$. For independent a, b the equality $S_{x, y, a} = S_{x, y, b}$ holds if x, y is a solution of system of linear equations $p_i x + q_i y = a, p_j x + q_j y = b$, where $q_i = \pm q_j$. In other words $R_3(x, y, z) \rightarrow z = (x - y)/(p_i - p_j) \vee z = (x + y)/(p_i + p_j)$ and in the case $z = (x + y)/(p_i + p_j)$ the table must contain the line $z = (x - y)/(p_i - p_j)$ as well. \square

Step 3. *The relation $z = \pm x \pm y$ is definable by R .*

Consider the relation $R_4(x, y, \Delta, z) \Leftarrow (\exists v, v')(R_3(z, \Delta, v) \wedge R_3(x, \Delta, v') \wedge R_3(v', y, v))$. In fact the $R_4(x, y, \Delta, z)$ is equivalent to disjunction of equations $z + a_i \Delta = x + a_j \Delta + a_k y$ for nonzero x, y and $\Delta \notin \mathcal{V}(\{x, y, z, \})$. Note that the table of R_4 is a subset of the table of R_3 .

We consider two cases.

Case 1. The table of R_4 contains lines $z = x + q_i y$ only.

Let N be a number of lines in the table of R_4 and denote by $W(y, z) \Leftarrow (\exists_{<N} x)(x \neq 0 \wedge R_4(x, y, z))$. It is easy to note that $W(y, z) \Leftrightarrow z = q_i y$.

Now we define $R_5(x, y, z) \Leftarrow (\forall y_1)(W(y_1, y) \Rightarrow R_4(x, y_1, z))$ and show that $R_5(x, y, z) \Leftrightarrow z = x + y$ or $R_5(x, y, z) \Leftrightarrow z = x \pm y$.

The sentence $R_5(x, y, z)$ holds iff for any q_i there is q_j such that $z = x + (q_i/q_j)y$. It follows that $q_i = \pm q_j$.

Case 2. The table of R_4 contains lines $z = x + q_i y$ and lines $z = -x + q_j y$.

According item (ii) Step 2 the relation $x = \pm y$ is definable by R_4 , so we can consider the relation $R_5(x, y, z) \Leftrightarrow R_4(\pm x, \pm y, z)$ the table of which is $z = \pm x \pm q_i y$. Denote by $W(y, z) \Leftrightarrow R_5(z, z, y) \wedge y \neq \vec{0}$. It is easy to see, that $W(y, z) \Leftrightarrow y = 2z/q_i$. We use the same arguments as in case 1 to show that $(\forall y_1)(W(y, y_1) \rightarrow R_5(x, y_1, z)) \Leftrightarrow z = \pm x \pm 2y$. We can now define $z = \pm 2y, y \neq \vec{0}$ as $z \neq \vec{0} \wedge (\exists_{=2} x)(x \neq \vec{0} \wedge z = \pm x \pm 2y)$ hence $z = \pm x \pm y$ is definable. \square

By the symbol $S^\pm(\mathcal{M})$ we denote the group of permutations of \mathcal{M} which satisfy condition $\sigma(x) = \pm x, x \in \mathcal{M}$. Symbol $GL^\pm(\mathcal{M})$ denotes $\langle GL(\mathcal{M}), S^\pm(\mathcal{M}) \rangle$.

Statement 5. $GL^\pm(\mathcal{M})$ is the group of automorphisms of the relation $z = \pm x \pm y$

Proof. (i) It is obvious that $\sigma \in GL^\pm(\mathcal{M}) \Rightarrow \sigma$ preserves $z = \pm x \pm y$.

(ii) Suppose, that σ preserves $z = \pm x \pm y$. Note that $\sigma(px) = \pm p\sigma(x)$. Choose v_1, \dots, v_n, \dots a basis of \mathcal{M} . We can suppose that $\sigma(v_i) = v_i$, so $\sigma(\sum p_i v_i) = \sum \pm p_i v_i$. Consider a sequence $w_k = \sum_{i=1}^k v_i$ and find a such permutation $\sigma' \in GL(\mathcal{M}), \sigma'(v_i) = \pm v_i$, that for any m there is an $n > m, \sigma^*(w_n) = w_n$ where $\sigma^* = \sigma' \circ \sigma$. Take an item $a \in \mathcal{M}$ and show that $\sigma^*(a) = \pm a$. Select such n that $i > n \rightarrow (a)_i = 0$. Then $\sigma^*(w_n + a) = \sum_{i=1}^n \pm(1 + p_i)v_i = \pm \sigma^*(w_n) \pm \sigma^*(a) = \pm w_n \pm \sigma^*(a)$, so $\sigma^*(a) = \pm a$. \square

Corollary 3. If R is nonaffine relation, then $z = \pm x \pm y$ is definable by R . If $GL^\pm(\mathcal{M})$ preserves R then R is equivalent to $z = \pm x \pm y$.

We postpone the proof of statement 4 till the statement 7 where more general situation is discussed.

4. RELATIONS WITH MORE THEN 3 ARGUMENTS.

We prove

Statement 6. If a relation $R(x_1, \dots, x_n)$ is 2-undefinable, then a triadic 2-undefinable relation is definable by R .

The notion of table can be generalized for a relation with more than 3 arguments.

First we prove

Lemma 7. If a relation $R(x_1, \dots, x_n)$ is 2-undefinable, then by R can be defined a relation $R'(x_1, \dots, x_k)$ which table contains a line $x_k = \sum p_i x_i$, where $p_1, p_2 \neq 0$.

Proof. We use the notion *rank* from the statement 1.

Proof by induction on number of arguments of R , on rank k of R and number of corresponding tuples x_{i_1}, \dots, x_{i_k} of length k .

Let a rank of R be $k < n$, we can suppose that $R(a_1, \dots, a_k, b_1, \dots, b_{n-k})$ holds for independent \bar{a} . Note that because a rank of R is k , there are only finite numbers such tuples \bar{b}' , that $R(\bar{a}, \bar{b}')$ holds.

We consider 2 cases:

(A) There is a tuple \bar{b}' , such that $R(\bar{a}, \bar{b}')$ holds and some $b'_j \notin \bigcup_{i=1}^k l(\vec{0}, a_i)$.

Then it's easy to note that the relation $R'(x_1, \dots, x_k, y_j) \Leftarrow (\exists y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{n-k}) R(\bar{x}, \bar{y})$ satisfies the required condition.

(B) For any tuple \bar{b} if $R(\bar{a}, \bar{b})$ holds then each b_j belongs to a line $l(\vec{0}, a_i)$.

In this case we will construct a definable by R 2-definable relation $Q(\bar{x}, \bar{y})$, such that for independent \bar{a} holds $(\forall \bar{y})(R(\bar{a}, \bar{y}) \equiv Q(\bar{a}, \bar{y}))$.

Next we consider relations $P(x_1, \dots, x_k) \Leftarrow (\forall \bar{y})(R(\bar{x}, \bar{y}) \equiv Q(\bar{x}, \bar{y}))$, $R_1(\bar{x}, \bar{y}) \Leftarrow P(\bar{x}) \wedge R(\bar{x}, \bar{y})$; $R_2(\bar{x}, \bar{y}) \Leftarrow \neg P(\bar{x}) \wedge R(\bar{x}, \bar{y})$. Note, that $R \equiv R_1 \vee R_2$, so one of them has to be 2-undefinable.

From the definition of P follows that $R_1(\bar{x}, \bar{y}) \equiv P(\bar{x}) \wedge Q(\bar{x}, \bar{y})$, so if P (with a less number of arguments than n) is 2-definable, then R_1 is 2-definable as well. From the other hand from properties of Q follows that $P(\bar{a})$ holds for independent \bar{a} , so $(\forall \bar{y}) \neg R_2(\bar{a}, \bar{y})$ holds. So the number of independent tuples of length k for R_2 is less than for R .

So it remains to construct such relation $Q(\bar{x}, \bar{y})$, that

- (i) for independent \bar{a} holds $(\forall \bar{y})(R(\bar{a}, \bar{y}) \equiv Q(\bar{a}, \bar{y}))$.
- (ii) Q is 2-definable.
- (iii) Q is definable by R .

Let us remind that we consider the case when each b_j belongs to a line $l(\vec{0}, a_i)$. For any tuple $s = \langle m_1, \dots, m_{n-k} \rangle$, $m_j \leq k$ we will define a individual relation Q_s , describing the situation when $b_j \in l(\vec{0}, a_{m_j})$ and set $Q \Leftarrow \bigvee_s Q_s$.

So we fix a $s = \langle m_1, \dots, m_{n-k} \rangle$, $m_j \leq k$. For each $j \leq n-k$ define a relation

$$S_j(x_{m_j}, y_j) \Leftarrow (\exists \succ_M x_1, \dots, x_{m_j-1}, x_{m_j+1}, \dots, x_k) (\exists y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{n-k}) R(\bar{x}, \bar{y})$$

It's clear, that for independent $a \neq \vec{0}$ the set $\{y | S_j(a, y)\}$ is finite, and $\{y | S_j(a, y)\} \subset l(\vec{0}, a)$, and $R(\bar{a}, \bar{b}) \rightarrow S_j(a_{m_j}, b_j)$ for independent \bar{a} if $b_j \in l(\vec{0}, a_{m_j})$. So $S_j(x, y) \equiv y = \alpha_{j,1}x \vee \dots \vee y = \alpha_{j,n_j}x$ for a finite set $\{\alpha_{j,1}, \dots, \alpha_{j,n_j}\}$ of rational numbers. Note that $\vec{0}$ is definable by S_j . Without loss of generality we can suppose that $R(\bar{z})$ is false, if at least one argument is equal to $\vec{0}$.

We say that a string $t = \langle \alpha_1^t, \dots, \alpha_{n-k}^t \rangle$ of rational numbers is *regular*, if for some (for any) independent \bar{a} holds $R(\bar{a}, \alpha_1^t \cdot a_{m_1}, \dots, \alpha_{n-k}^t \cdot a_{m_{n-k}})$. Define $T_s(\bar{x}, \bar{y}) \Leftarrow \bigvee_t y_1 = \alpha_1^t \cdot x_{m_1} \wedge \dots \wedge y_{n-k} = \alpha_{n-k}^t \cdot x_{m_{n-k}}$, the disjunction contains all regular strings t . Conditions (i) and (ii) for T_s follows from the definition immediately.

Prove now that T_s is definable by R . First for each $j \leq n-k$ we define an equivalence E_j . Without loss of generality we describe the equivalence E_1 . Let m_1, \dots, m_l be the list of all numbers $m_i \leq n-k$ such that $s_{m_i} = 1$. We may suppose that they are $1, \dots, l$. We say that 2 tuples a, b_1, \dots, b_l and a', b'_1, \dots, b'_l are equal ($E_1(a, \bar{b}, a', \bar{b}')$)

holds) if

$$\bigwedge_{i \leq l} (S_i(a, b_i) \wedge S_i(a', b'_i)) \\ \wedge (\exists_{>M} x_2, \dots, x_k) (\forall y_{l+1}, \dots, y_{n-k}) (R(a, x_2, \dots, x_k, b_1, \dots, b_l, y_{l+1}, \dots, y_{n-k}) \equiv \\ R(a', x_2, \dots, x_k, b'_1, \dots, b'_l, y_{l+1}, \dots, y_{n-k}))$$

Note

(*) that if $\{v, u\}$ are independent, then $E_1(v, p_1 \cdot v, \dots, p_l \cdot v, u, p_1 \cdot v, \dots, p_l \cdot v)$ holds for any rationals $p_i \neq 0$.

Show that $T_s(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ is equivalent to

$$(2) \quad (\exists_{>M} x'_1, \dots, x'_k) (\forall y'_1, \dots, y'_{n-k}) \\ \left(\bigwedge_{i=1}^k E_i(x_i, \bar{y}_i, x'_i, \bar{y}'_i) \rightarrow R(x'_1, \dots, x'_k, y'_1, \dots, y'_{n-k}) \right)$$

Suppose (2) holds. We need to show, that for independent \bar{a} holds $R(\bar{a}, \alpha_1 \cdot a_{m_1}, \dots, \alpha_{n-k} \cdot a_{m_{n-k}})$, where $\alpha_i \cdot x_{m_i} = y_i$. It immediately follows from (2) and note (*).

To the opposite. Choose a regular string $\langle \alpha_1, \dots, \alpha_{n-k} \rangle$ and set $y_i = \alpha_i \cdot x_{m_i}$. We need to show that (2) holds. By the definition of regular string there is an independent tuple \bar{a} , such that $R(\bar{a}, \bar{b})$ holds where $b_i = \alpha_i \cdot a_{m_i}$. Select a tuple \bar{a}' , such that $\bar{a} \cup \bar{a}' \cup \bar{x}$ is independent. Prove that if $E_i(x_{m_i}, \bar{y}_i, a'_{m_i}, \bar{b}_i)$ holds for all i then $R(\bar{a}', \bar{b}')$ holds. $E_1(a_1, b_1, \dots, b_l, a'_1, b'_1, \dots, b_l)$ holds because $E_1(x_1, y_1, \dots, y_l, a'_1, b'_1, \dots, b'_l)$ and $E_1(x_1, y_1, \dots, y_l, a_1, b_1, \dots, b_l)$. By the definition of E_1 follows $R(a'_1, a_2, \dots, a_k, b'_1, \dots, b'_l, b_2, \dots, b_{n-k}) \equiv R(\bar{a}, \bar{b})$. Continuing this procedure we get $R(\bar{a}', \bar{b}')$. \square

Lemma 8. *If the table of relation $R(x_1, \dots, x_n, z)$ contains a line $\sum_{i=1}^n p_i x_i = z$ where $p_1, p_2 \neq 0$ then there is definable by R 2-undefinable triadic relation.*

Proof. Due to lemma 4 we suppose, that $p_1^2 + p_2 \neq 0$.

Consider a relation $R'(x_1, \Delta, x_3, \dots, x_n, z) \equiv (\exists v)(R(x_1, \Delta, x_3, \dots, x_n, v) \wedge R(v, x_2, \dots, x_n, z))$ which contains for independent $\{\Delta, x_1, x_3, \dots, x_n\}$ and $\{\Delta, x_2, \dots, x_n\}$ the line $z = p_1^2 x_1 + p_1 p_2 \Delta + p_2 x_2 + p_3(1 + p_1)x_3 + \dots + p_n(1 + p_1)x_n$. So a relation $R^*(x_1, x_2, z) \equiv (\exists_{>M} \Delta, x_3, \dots, x_n) (\exists w)(R'(\Delta, x_1, \dots, x_n, w) \wedge R'(\Delta, z, z, x_3, \dots, x_n, w))$ holds for independent $\{x_1, x_2\}$ when $(p^2 + q)z = p^2 x_1 + p_2 x_2$ and there are a finite number of such z that $R^*(x_1, x_2, z)$ holds. \square

So if by 2-undefinable relation R an affine ternary relation is definable, then (according to lemma 1) the automorphism group of R is a subgroup of $AGL(\mathcal{M})$ and coincide with this group if $\bar{0}$ is not definable by R , otherwise it coincide with $GL(\mathcal{M})$.

If by 2-undefinable relation R a nonaffine ternary relation is definable, then the relation $z = \pm x \pm y$ is definable as well (lemma 6). We are going to prove, that if the group $GL^\pm(\mathcal{M})$ does not preserves R then or $z = x + y$ is definable by R

or R is equivalent to signature $\{z = \pm x \pm y, R^*(\bar{x})\}$ for some 2-definable relation R^* .

We start with

Lemma 9. *Suppose that for a relation $R(\bar{x}, z)$, a tuple $\bar{r} \in \mathbb{Q}, r_i \neq 0$ and for any independent tuple $\bar{a} \in \mathcal{M}$ holds*

$$R(\bar{a}, z) \Leftrightarrow z = r_1 a_1 \pm r_2 x_2 \cdots \pm r_n a_n$$

Then $z = x + y$ is definable by R .

Proof. Let us remind that the relations $y = -x$ and $y = \pm r x$ for any $r \in \mathbb{Q}$ are definable by $z = \pm x \pm y$ and hence by R .

Define the relation $R_1(x, y, z) \Leftrightarrow (\exists x_2, \dots, x_n)(y = \pm r_2 x_2 \cdots \pm r_n x_n \wedge z = \pm r_1 x \pm y \wedge R(x, x_2, \dots, x_n, z))$ and note that for independent x, y holds $R_1(x, y, z) \Leftrightarrow z = r_1 x \pm y$. Next define $R_2(x, y, \Delta, z) \Leftrightarrow (\exists u, v)(R_1(z, \Delta, u) \wedge R_1(y, \Delta, v) \wedge R_1(x, v, u))$. If $\Delta \notin \mathcal{V}(\{x, y, z\})$ then $R_2(x, y, \Delta, z) \Leftrightarrow r_1 z \pm \Delta = r_1 x \pm r_1 y \pm \Delta$ hence $(\exists_{>M} \Delta) R_2(x, y, \Delta, z) \Leftrightarrow z = x \pm y$ for sufficiently large M and $x, y, z \neq 0$. Therefore $z = x + y \Leftrightarrow ((x \neq y \wedge x \neq \vec{0} \wedge y \neq 0 \rightarrow (z = x \pm y \wedge z = y \pm x)) \wedge (x = y \wedge x \neq \vec{0} \rightarrow (z = x \pm x \wedge z \neq \vec{0})) \wedge (x = \vec{0} \rightarrow z = y) \wedge (y = \vec{0} \rightarrow z = x))$ \square

Lemma 10. *Suppose that for a relation $R(\bar{t}, x_1, \dots, x_n, z)$ where $n > 1$, some $m > 0$, any tuple $\bar{r} \in \mathbb{Q}, r_i \neq 0$, any parameters $\bar{t} \in \mathcal{M}$, and for any m -independent tuple $\bar{a} \in \mathcal{M}$ holds*

$$R(\bar{t}, \bar{a}, z) \Leftrightarrow \bigvee_{s \in S_{\bar{t}}} s(i) r_i a_i \text{ where } S_{\bar{t}} \subset \{-1, 1\}^n$$

and $0 < |S_{\bar{t}}| < 2^n$ for some $\bar{t} \in \mathcal{M}$.

Then $z = x + y$ is definable by $\Sigma = \{z = \pm x \pm y, R\}$.

Proof. Note, that m -independency is definable by $z = \pm x \pm y$.

We may suppose that $(\forall \bar{t})(\exists z)(R(\bar{t}, \bar{a}, z) \rightarrow |S_{\bar{t}}| = k)$ holds for any m -independent \bar{a} and some $0 < k < 2^n$, otherwise we consider the relation $R(\bar{t}, \bar{x}, z) \wedge |S_{\bar{t}}| = k$ for an appropriate k .

For any m -independent \bar{x} and z we denote by $W_{\bar{x}, z}$ the set $\{z' | (\exists u)(u = \pm r_1 x_1 \cdots \pm r_{n-1} x_{n-1} \wedge z = \pm u \pm r_n x_n \wedge z' = \pm u \pm r_n x_n)\}$. If $R(\bar{t}, \bar{x}, z)$ holds, then $z' \in W_{\bar{x}, z} \Leftrightarrow (z = \sum_{i=1}^n s(i) r_i a_i, z' = \pm \sum_{i=1}^n s(i) r_i a_i \pm r_n x_n \text{ for some } s \in S_{\bar{t}})$. Note, that $z \in W_{\bar{x}, z}$ and $|W_{\bar{x}, z}| = 4$. By $W'_{\bar{t}, \bar{x}, z}$ we denote $\{z' | R(\bar{t}, \bar{x}, z'), z' \in W_{\bar{x}, z}\}$.

Now for any m -independent \bar{x} and z such that $R(\bar{t}, \bar{a}, z)$ we define the *type* $(T(\bar{t}, \bar{a}, z))$ – a natural number $i, i < 7$, the relation $T(\bar{t}, \bar{x}, z) = i$ will be definable by Σ for each i .

$$(i) \ T(\bar{t}, \bar{x}, z) = 1 \Leftrightarrow |W'_{\bar{t}, \bar{x}, z}| = 4$$

$$(ii) \ T(\bar{t}, \bar{x}, z) = 2 \Leftrightarrow |W'_{\bar{t}, \bar{x}, z}| = 1$$

$$(iii) \ T(\bar{t}, \bar{x}, z) = 3 \Leftrightarrow |W'_{\bar{t}, \bar{x}, z}| = 3$$

$$(iv) \ T(\bar{t}, \bar{x}, z) = 4 \Leftrightarrow (|W'_{\bar{t}, \bar{x}, z}| = 2 \wedge R(\bar{t}, \bar{x}, -z))$$

$$(v) \quad T(\bar{t}, \bar{x}, z) = 5 \Leftrightarrow (|W'_{\bar{t}, \bar{x}, z}| = 2 \wedge \neg R(\bar{t}, \bar{x}, -z) \wedge R(\bar{t}, x_1, \dots, x_{n-1}, -x_n, z))$$

$$(vi) \quad T(\bar{t}, \bar{x}, z) = 6 \Leftrightarrow (|W'_{\bar{t}, \bar{x}, z}| = 2 \wedge \neg R(\bar{t}, \bar{x}, -z) \wedge \neg R(\bar{t}, x_1, \dots, x_{n-1}, -x_n, z))$$

By T_R we denote the minimal i such that $(\exists \bar{t}, \bar{x}, z)((\{x_1, \dots, x_n\} \text{ is } m\text{-independent}) \wedge R(\bar{t}, \bar{x}, z) \wedge T(\bar{t}, \bar{x}, z) = i)$ and denote $R_1(\bar{t}, \bar{x}, z) \Leftrightarrow R(\bar{t}, \bar{x}, z) \wedge T(\bar{t}, \bar{x}, z) = T_R$. Note that for some $S_{1, \bar{t}} \subset S_{\bar{t}}, S_{1, \bar{t}} \neq \emptyset$ and any m -independent \bar{a} holds

$$R_1(\bar{t}, \bar{a}, z) \Leftrightarrow \bigvee_{s \in S_{1, \bar{t}}} z = \sum_{i=1}^n s(i) r_i a_i$$

For a function $s \in \{-1, 1\}^k, \alpha = \pm 1$ we denote by $\langle s, \alpha \rangle \in \{-1, 1\}^{k+1}$ extension s on $\{1, \dots, k+1\}$ such that $\langle s, \alpha \rangle(i) = s(i), i < k+1, \langle s, \alpha \rangle(k+1) = \alpha$ by s^* we denote the initial segment of s i.e. $s = \langle s^*, s(k) \rangle$. If $S \subset \{-1, 1\}^k$ then $S^* = \{s^* | s \in S\}$.

Proof by induction on n .

The basis of the induction, case $n=2$, will be considered later.

For $n > 2$ we will consider all 6 cases:

(i) $T_R = 1$: if $s \in S_{1, \bar{t}}^*$ then all 4 strings $\langle \pm s, \pm 1 \rangle$ belong to $S_{1, \bar{t}}$ as well. Define the relation $R_2(\bar{t}, x_1, \dots, x_{n-1}, z) \Leftrightarrow z = \sum_{i=1}^{n-1} \pm r_i a_i \wedge (\exists x_n, z')((\{x_1, \dots, x_n\} \text{ is } m\text{-independent}) \wedge R_1(\bar{t}, \bar{x}, z') \wedge z' = \pm z \pm r_n a_n)$. For m -independent a_1, \dots, a_{n-1} holds $R_2(\bar{t}, \bar{a}, z) \Leftrightarrow (z = \sum_{i=1}^{n-1} s(i) r_i a_i \text{ for some } s \in S_{1, \bar{t}}^*)$. Because $|S_{1, \bar{t}}^*| = |S_{1, \bar{t}}|/2$ we can use induction.

(ii) $T_R = 2$: if $s \in S_{1, \bar{t}}^*$ then $-s \notin S_{1, \bar{t}}^*$ and $\langle s, 1 \rangle \in S_{1, \bar{t}} \Leftrightarrow \langle s, -1 \rangle \notin S_{1, \bar{t}}$. Consider the relation $R_2(\bar{t}, \bar{x}, z, u) \Leftrightarrow R_1(\bar{t}, \bar{x}, z) \wedge u = \sum_{i=1}^{n-1} \pm r_i (1 + r_n) x_i \pm r_n^2 x_n \wedge R_1(\bar{t}, x_1, \dots, x_{n-1}, z, u)$. Let a tuple \bar{a} be m -independent, $u = \sum_{i=1}^{n-1} s_1(i) r_i a_i + s_1(n) r_n (\sum_{i=1}^n s_2(i) r_i a_i)$ for some $s_1, s_2 \in S_1$. Then $u = \sum_{i=1}^{n-1} \pm r_i (1 + r_n) x_i \pm r_n^2 x_n$ if $s_1^* = s_2^*$ and $s_1(n) = 1$ or $s_1^* = -s_2^*$ and $s_1(n) = -1$. From $s \in S_{1, \bar{t}}^* \rightarrow -s \notin S_{1, \bar{t}}^*$ follows that $(\exists u) R_2(\bar{t}, \bar{x}, z, u) \Rightarrow z = \sum_{i=1}^{n-1} s(i) r_i x_i + r_n x_n$ for m -independent \bar{x} . If the relation $(\exists u) R_2(\bar{t}, \bar{x}, z, u)$ is nonempty for some parameters \bar{t} and m -independent \bar{x} then we can use the Lemma 9 renaming x_n to x . If $(\exists u) R_2(\bar{t}, \bar{x}, y, z)$ is empty for any parameters \bar{t} then $s(n) = -1$ for any $s \in S_{1, \bar{t}}$, hence we can use Lemma 9 for the relation R_1 .

(iii) $T_R = 3$: if $s \in S_{1, \bar{t}}^*$ then there is only one item $s' = \langle \pm s, \pm 1 \rangle, s' \notin S_{1, \bar{t}}$. Define $R_3(\bar{t}, \bar{x}, z) \Leftrightarrow (\exists z')(R(\bar{t}, \bar{x}, z') \wedge z \in W_{\bar{x}, z'} \setminus W'_{\bar{t}, \bar{x}, z'})$. The relation R_3 meets condition of the case (ii).

(iv) $T_R = 4$: $\langle s, 1 \rangle \in S_{1, \bar{t}} \Leftrightarrow \langle -s, -1 \rangle \in S_{1, \bar{t}}$. Consider the relation $R_2(\bar{t}, \bar{x}, z, u)$ from the case (ii). Just as in that case we see, that $(\exists u) R_2(\bar{t}, \bar{x}, z, u) \Rightarrow z = \sum_{i=1}^{n-1} \pm r_i x_i + r_n x_n$ for m -independent \bar{x} , so we can use the Lemma 9 for the relation $(\exists u) R_2(\bar{t}, \bar{x}, z, u)$.

(v) $T_R = 5$: if $s \in S_{1, \bar{t}}^*$ then $\langle s, 1 \rangle, \langle s, -1 \rangle \in S_{1, \bar{t}}, -s \notin S_{1, \bar{t}}^*$. Consider the relation $R_3(x_1, \dots, x_{n-1}, z, x_n)$ and note that in this case it meets conditions of case (iv).

(vi) $T_R = 6$: if $s \in S_{1, \bar{t}}^*$ then $-s \in S_{1, \bar{t}}^*$ and $\langle s, 1 \rangle \in S_{1, \bar{t}}^* \Leftrightarrow \langle -s, 1 \rangle \in S_{1, \bar{t}}$. Consider the relation $R_2(\bar{t}, \bar{x}, z, u)$ from the case (ii). It is easy to check that in this case

if $R_2(\bar{t}, \bar{x}, z, u)$ holds then $u = \sum_{i=1}^{n-1} \pm r_i(1 + r_n)x_i + r_n^2 x_n$ so so we can use the Lemma 9 for the relation $R_3(\bar{t}, \bar{x}, u) \equiv (\exists z)R_2(\bar{t}, \bar{x}, z, u)$.

Basis of the induction: $n = 2$. The same as $n > 2$ except the case (i) because $T_R < 4$.

□

Statement 7. *Suppose that R is 2-undefinable nonaffine relation. Then or R is equivalent to $z = x + y$ or R is equivalent to $\{z = \pm x \pm y, R^*\}$ for some 2-definable relations R^* .*

Proof. We know, that $z = \pm x \pm y$ is definable by R , so is the relation $y = -x$ and relations $y = \sum_i \pm r_i x_i$ for any $r_i \in \mathbb{Q}$, so the m -independency is definable for any m as well.

Let m be such number that $\neg R(\bar{a})$ for any m -independent \bar{a} .

We prove by induction of n – numbers of arguments of R .

Let $s \subset \{1, \dots, n\}, s \neq \emptyset, s \neq \{1, \dots, n\}$ and $p = \{r_{i,j} | i \notin s, j \in s, r_{i,j} = l/k \text{ for some } |k|, |l| < m\}$. Denote by $R_{s,p}(\bar{x})$ the statement $(\{x_i | i \in s\} \text{ is } m\text{-independent} \wedge \bigwedge_{i \notin s} x_i = \sum_{j \in s} \pm r_{i,j} x_j \wedge R(\bar{x}))$. The relation $R_{s,p}(\bar{x})$ is definable by R and $R(\bar{x}) \equiv \bigvee_{s,p} R_{s,p}(\bar{x})$.

We prove the statement for each $R_{s,p}(\bar{x})$, without loss of generality we suppose that $s = \{1, \dots, k\}$.

Consider different cases:

(i) For any $i > k$ there is only one $l_i \leq k$ such that $r_{i,l_i} \neq 0$. In other words $R_{s,p}(\bar{x})$ is equivalent to $(\{x_1, \dots, x_k\} \text{ is } m\text{-independent} \wedge \bigwedge_{i > k} x_i = \pm r_{l_i} x_{l_i} \wedge R(\bar{x}))$. Then $R_{s,p}(\bar{x})$ is equivalent to $(\{x_1, \dots, x_k\} \text{ is } m\text{-independent} \wedge \bigvee_{\bar{p} \in A} \bigwedge_{i > k} x_i = p_i x_{l_i})$ where $A = \{\bar{p} | p_i = \pm r_{l_i}, R(a_1, \dots, a_k, p_{k+1}, \dots, p_n, x_{l_n}) \text{ holds for independent } \{a_1, \dots, a_k\}\}$.

(ii) $r_{n,1}, \dots, r_{n,m} \neq 0, r_{n,m+1}, \dots, r_{n,k} = 0$, where $1 < m \leq k$.

Consider the relation $R_{s,p}(x_1, \dots, x_k, t_1, \dots, t_{n-k-1}, z)$. If for any m -independent $\{a_1, \dots, a_k\}$ holds $(\exists \bar{t})(0 < |z|_{R_{s,p}(a_1, \dots, a_k, t_1, \dots, t_{n-k-1}, z)}| < 2^m)$ then, due to Lemma 10, the relation $z = x + y$ is definable by $R_{s,p}$ and hence by R .

If for any m -independent $\{a_1, \dots, a_k\}$ holds

$$(\forall \bar{t})((\exists z)R_{s,p}(a_1, \dots, a_k, t_1, \dots, t_{n-k-1}, z) \rightarrow |z|_{R_{s,p}(a_1, \dots, a_k, t_1, \dots, t_{n-k-1}, z)}| = 2^m)$$

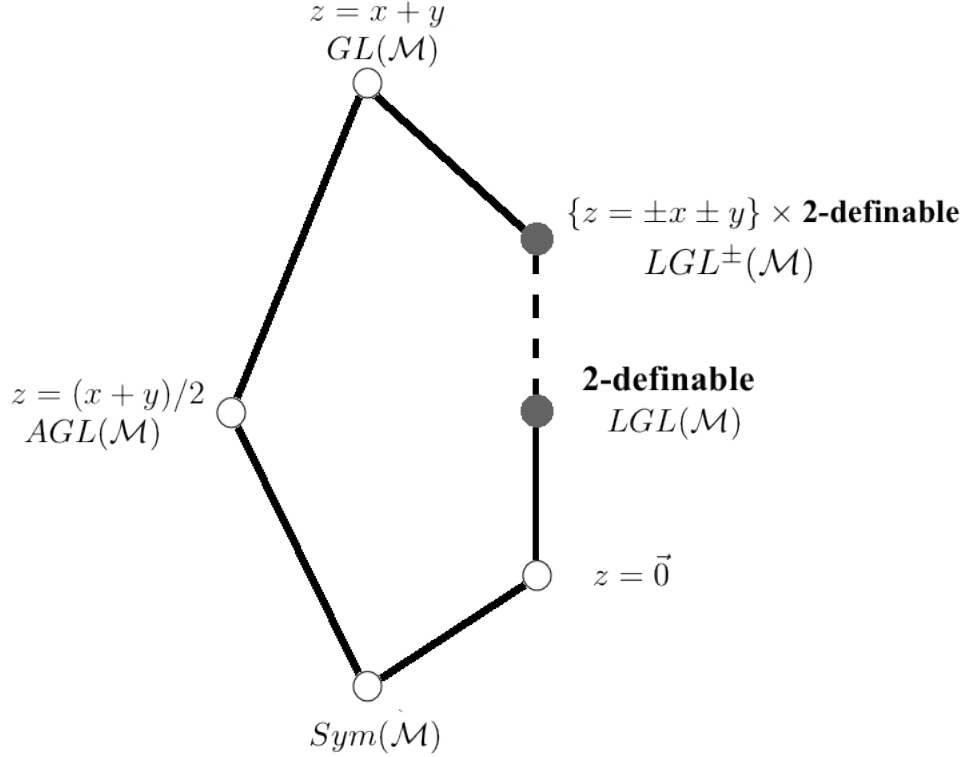
then

$$R_{s,p}(x_1, \dots, x_k, t_1, \dots, t_{n-k-1}, z) \equiv (\exists z)R_{s,p}(x_1, \dots, x_k, t_1, \dots, t_{n-k-1}, z) \wedge z = \sum_{i=1}^m \pm r_{n,i} x_k$$

so we can use induction for the relation $(\exists z)R_{s,p}(x_1, \dots, x_k, t_1, \dots, t_{n-k-1}, z)$ □

The group of permutations, preserving all relations of form $\{z = \pm x \pm y, y = p_1x, \dots, y = p_nx\}$ we denote $LGL^\pm(\mathcal{M})$: beside $GL(\mathcal{M})$ it contains such permutations σ that for each line l passing through $\vec{0}$ or $\sigma(x) = x, x \in l$ or $\sigma(x) = -x, x \in l$.

5. SUMMARY.



Comments:

- (i) Solid gray vertex denotes a family of relations
- (ii) Dotted line means that members of one family is defined by members of another one.

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